INSTRUMENTATION ENGINEERING

Electricity and Magnetism

Comprehensive Theory
with Solved Examples *and* Practice Questions

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Electricity and Magnetism

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Electricity and Magnetism

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CHAPTER

1

Vector Analysis

1.1 INTRODUCTION

Vector analysis is a concise language or mathematical shorthand which greatly facilitates the analysis of electric and magnetic fields. The quantities of interest appearing in the study of EM theory can almost be classified as either a scalar or a vector.

Quantities that can be described by a magnitude alone are called **scalars**. Distance, temperature, mass etc. are examples of scalar quantities. Other quantities, called **vectors**, require both a magnitude and a direction to fully characterize them. Examples of vector quantities include velocity, force, acceleration etc.

In electromagnetics, we frequently use the concept of a **field**. A field is a function that assigns a particular physical quantity to every point in a region. In general, a field varies with both position and time. There are scalar fields and vector fields. Temperature distribution in a room and electric potential are examples of scalar fields. Electric field and magnetic flux density are examples of vector fields.

Note: Vectors are denoted by an arrow over a letter ($\vec{A}$) and scalars are denoted by simple letter (A).

1.1.1 Unit Vector

A unit vector $\hat{a}_A$ along $\vec{A}$ is defined as a vector whose magnitude is unity (*i.e.*, 1) and its direction is along $\vec{A}$, that is

$$\hat{a}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A} \quad ...(1.1)$$

Thus we can write $\vec{A}$ as

$$\vec{A} = A\,\hat{a}_A = |\vec{A}|\hat{a}_A \quad ...(1.2)$$

Remember: Any vector can be written as product of its magnitude and its unit vector.

1.1.2 Vector Addition and Subtraction

Two vectors $\vec{A}$ and $\vec{B}$ can be added together to give another vector $\vec{C}$; that is,

$$\vec{C} = \vec{A} + \vec{B} \quad ...(1.3)$$

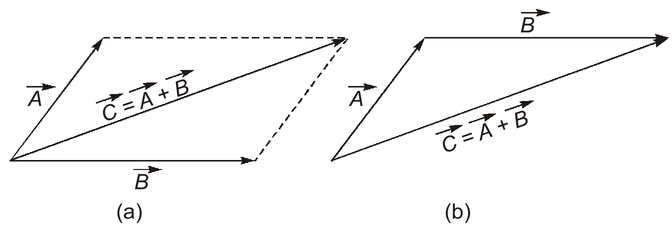


***Fig.:** Vector addition (a) parallelogram rule, (b) head-to-tail rule.*

NOTE

- $\vec{A}+\vec{B}=\vec{B}+\vec{A}$ (Commutative law).
- $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$ (Associative law)

Vector subtraction is similarly carried out as

$$\vec{D} = \vec{A}-\vec{B}=\vec{A}+(-\vec{B}) \quad ...(1.4)$$

REMEMBER Graphically, vector addition and subtraction are obtained by either the parallelogram rule or the head-to-tail rule.

NOTE

- $k(\vec{A}+\vec{B})=k\vec{A}+k\vec{B}$ (Distributive law)
- $\dfrac{\vec{A}+\vec{B}}{k}=\dfrac{1}{k}\vec{A}+\dfrac{1}{k}\vec{B}$

1.1.3 Position and Distance Vectors:

A point P in Cartesian coordinates may be represented by (x, y, z).

The position vector $\vec{r}_p$ (or radius vector) of point P is defined as the directed distance from origin O to P.

$$\vec{r}_p = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z \quad ...(1.5)$$

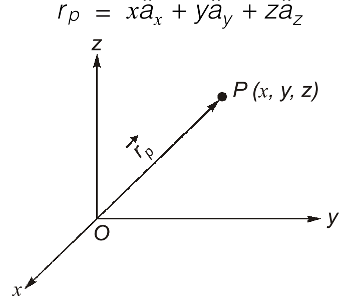


***Fig.:** Illustration of position vector* $\vec{r}_p = x\hat{a}_x + \hat{y}a_y + z\hat{a}_z$

The distance vector is the displacement from one point to another.

Consider point P with position vector $\vec{r}_p$ and point Q with position vector $\vec{r}_q$. The displacement from P to Q is written as

$$\vec{R}_{PQ} = \vec{r}_q - \vec{r}_p \quad ...(1.6)$$

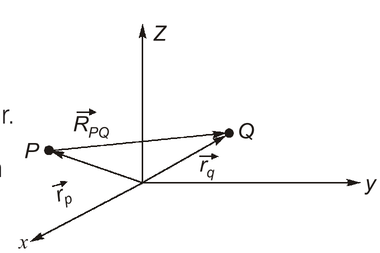


***Fig.:** Vector distance* $\vec{R}_{PQ}$

EXAMPLE : 1.1 Point P and Q are located at (0, 2, 4) and (–3, 1, 5). Calculate

(a) The position vector P

(b) The distance vector from P to Q

(c) The distance between P and Q

(d) A vector parallel to PQ with magnitude of 10.

Solution:

(a) $$\vec{r}_p = 0\hat{a}_x + 2\hat{a}_y + 4\hat{a}_z = 2\hat{a}_y + 4\hat{a}_z$$

(b) $$\vec{R}_{PQ} = \vec{r}_q - \vec{r}_p = (-3, 1, 5) - (0, 2, 4) = (-3, -1, 1)$$
$$= -3\hat{a}_x - \hat{a}_y + \hat{a}_z$$

(c) The distance between P and Q is the magnitude of $\vec{R}_{PQ}$; that is

$$d = |\vec{R}_{PQ}| = \sqrt{9+1+1} = 3.317$$

(d) Let the required vector be $\vec{A}$, then

$$\vec{A} = A\hat{a}_A$$

where $A = 10$ is magnitude of $\vec{A}$

and $$\hat{a}_A = \frac{\vec{R}_{PQ}}{|\vec{R}_{PQ}|} = \pm\frac{(-3,-1,1)}{3.317}$$

then $$\vec{A} = \pm\frac{10(-3,-1,1)}{3.317} = =\pm(-9.045\,\hat{a}_x - 3.015\,\hat{a}_y + 3.015\,\hat{a}_z)$$

1.1.4 Vector Multiplication

When two vectors are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication.

1. Scalar (or dot) product : $\vec{A} \cdot \vec{B}$
2. Vector (or cross) product : $\vec{A} \times \vec{B}$

Multiplication of three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ can result in either

3. Scalar triple product : $\vec{A} \cdot (\vec{B} \times \vec{C})$
4. Vector triple product : $\vec{A} \times (\vec{B} \times \vec{C})$

1. Dot Product

The dot product, or the scalar product of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A} . \vec{B}$ is defined geometrically as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle between them.

Thus $$\vec{A} \cdot \vec{B} = A\,B\cos\theta_{AB} \quad ...(1.7)$$

Where θ_{AB} is the smaller angle between $\vec{A}$ and $\vec{B}$. The result of $\vec{A} . \vec{B}$ is called either the scalar product because it is scalar, or the dot product due to the dot sign.

If $\vec{A} = (A_x, A_y, A_z)$ and $\vec{B} = (B_x, B_y, B_z)$

then $$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \quad ...(1.8)$$

The dot product is commutative and distributive.

Note: Two vectors $\vec{A}$ and $\vec{B}$ are said to be orthogonal (or perpendicular) with each other if $\vec{A}\cdot\vec{B} = 0$

The dot product obeys the following:

Law	Expression	
Cumulative	$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$	...(1.9)
Distributive	$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}.\vec{C}$	...(1.10)
	$\vec{A}\cdot\vec{A} = \lvert\vec{A}\rvert^2 = \lvert A\rvert^2$	...(1.11)

Also note that:

$$\hat{a}_x\cdot\hat{a}_y = \hat{a}_y\cdot\hat{a}_z = \hat{a}_z\cdot\hat{a}_x = 0 \quad ...(1.12)$$

$$\hat{a}_x\cdot\hat{a}_x = \hat{a}_y\cdot\hat{a}_y = \hat{a}_z\cdot\hat{a}_z = 1 \quad ...(1.13)$$

2. **Cross Product:**

The cross product of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A}\times\vec{B}$, is a vector quantity whose magnitude is the area of the parallelepiped formed by $\vec{A}$ and $\vec{B}$ and is in the direction of advance of the right-handed screw as $\vec{A}$ is turned into $\vec{B}$.

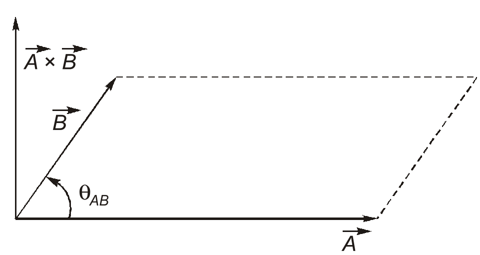


***Fig.:** The cross product of $\vec{A}$ and $\vec{B}$ is a vector with magnitude equal to the area of parallelogram and the direction as indicated*

Thus
$$\vec{A}\times\vec{B} = AB\sin\theta_{AB}\,\hat{a}_n \quad ...(1.14)$$

where $\hat{a}_n$ is a unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$.

The vector multiplication of equation (1.14) is called **cross product** due to the cross sign. It is also called **vector product** because the result is a vector.

If
$$\vec{A} = (A_x, A_y, A_z) \text{ and } B = (B_x, B_y, B_z) \text{ then :}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad ...(1.15)$$

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{a}_x + (A_zB_x - A_xB_z)\hat{a}_y + (A_xB_y - A_yB_x)\hat{a}_z \quad ...(1.16)$$

Which is obtained by 'crossing' terms in cyclic permutation, hence the name cross product.

Vector product is not commutative and not associative but vector product is distributive.

Note that the cross product has the following properties

1. It is not commutative:

$$\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A} \quad ...(1.17)$$

Note: $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$

2. It is not associative:

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C} \qquad ...(1.18)$$

3. It is distributive:

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C} \qquad ...(1.19)$$

Note: $\vec{A}\times\vec{A} = 0$

4. Also note that

$$\hat{a}_x\times\hat{a}_y = \hat{a}_z \qquad ...(1.20)$$

$$\hat{a}_y\times\hat{a}_z = \hat{a}_x \qquad ...(1.21)$$

$$\hat{a}_z\times\hat{a}_x = \hat{a}_y \qquad ...(1.22)$$

NOTE: If $\vec{A}\times\vec{B} = 0$, then $\sin\theta_{AB} = 0°$ or $180°$; this shows that $\vec{A}$ and $\vec{B}$ are parallel or antiparallel to each other

3. **Scalar Triple Product:**

Given three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, we define scalar triple product as,

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \qquad ...(1.23)$$

If $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$ and $\vec{C} = (C_x, C_y, C_z)$, then $\vec{A}\cdot(\vec{B}\times\vec{C})$ is the volume of a parallelepiped having $\vec{A}$, $\vec{B}$, and $\vec{C}$ as edges and is easily obtained by finding the determinant of the 3×3 matrix formed by $\vec{A}$, $\vec{B}$, and $\vec{C}$; that is

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \qquad ...(1.24)$$

Since the result of this vector multiplication is scalar these two equations are called the scalar triple product.

4. **Vector Triple Product:**

For vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, we define the vector triple product as

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B}) \qquad ...(1.25)$$

This is obtained using the "bac-cab" rule.

EXAMPLE : 1.2

Three field quantities are given by $\vec{P} = 2\hat{a}_x - \hat{a}_z$ and $\vec{Q} = 2\hat{a}_x - \hat{a}_y + 2\hat{a}_z$, $\vec{R} = 2\hat{a}_x - 3\hat{a}_y + \hat{a}_z$. Determine:

(a) $(\vec{P}+\vec{Q})\times(\vec{P}-\vec{Q})$ (b) $\vec{Q}\cdot(\vec{R}\times\vec{P})$ (c) $\vec{P}\cdot(\vec{Q}\times\vec{R})$

(d) $\sin\theta_{QR}$ (e) $\vec{P}\times(\vec{Q}\times\vec{R})$

(f) A unit vector perpendicular to both $\vec{Q}$ and $\vec{R}$

(g) The component of $\vec{P}$ along $\vec{Q}$

Solution:

(a) $$(\vec{P}+\vec{Q})\times(\vec{P}-\vec{Q}) = 2(\vec{Q}\times\vec{P})$$

$$= 2\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} = 2(1-0)\hat{a}_x + 2(4+2)\hat{a}_y + 2(0+2)\hat{a}_z = 2\hat{a}_x + 12\hat{a}_y + 4\hat{a}_z$$

(b) $$\vec{Q}\cdot(\vec{R}\times\vec{P}) = (2, -1, 2) \cdot \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= (2, -1, 2).(3, 4, 6) = 6 - 4 + 12 = 14$$

Alternatively: $$\vec{Q}\cdot(\vec{R}\times\vec{P}) = \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 14$$

(c) $$\vec{P}\cdot(\vec{Q}\times\vec{R}) = \vec{Q}\cdot(\vec{R}\times\vec{P}) = 14$$

(d) $$\sin\theta_{QR} = \frac{|\vec{Q}\times\vec{R}|}{|\vec{Q}|\,|\vec{R}|} = \frac{\sqrt{45}}{3\sqrt{14}} = 0.5976$$

(e) $$\vec{P}\times(\vec{Q}\times\vec{R}) = \vec{Q}(\vec{P}\cdot\vec{R}) - \vec{R}(\vec{P}\cdot\vec{Q})$$

$$= (2, -1, 2)(4+0-1) - (2, -3, 1)(4+0-2) = (2, 3, 4) = 2\hat{a}_x + 3\hat{a}_y + 4\hat{a}_z$$

(f) A unit vector perpendicular to both $\vec{Q}$ and $\vec{R}$ is given by

$$\hat{a}_n = \pm\frac{\vec{Q}\times\vec{R}}{|\vec{Q}\times\vec{R}|} = \frac{\pm(5, 2, -4)}{\sqrt{45}} = \pm(0.745, 0.298, -0.596)$$

$$\hat{a}_n = \pm(0.745\,\hat{a}_x + 0.298\,\hat{a}_y - 0.596\,\hat{a}_z)$$

Note that, $|\hat{a}_n| = 1,\ \hat{a}_n\cdot\vec{Q} = \hat{a}_n\cdot\vec{R} = 0$

The component of $\vec{P}$ along $\vec{Q}$ is

$$\vec{P}_Q = |\vec{P}|\cos\theta_{PQ}\,\hat{a}_Q = (\vec{P}\cdot\hat{a}_Q)\hat{a}_Q = \frac{(\vec{P}.\vec{Q})\vec{Q}}{|\vec{Q}|^2} = \frac{(4+0-2)(2,-1,2)}{(4+1+4)}$$

$$= \frac{2}{9}(2,-1,2) = 0.4444\,\hat{a}_x - 0.2222\,\hat{a}_y + 0.4444\,\hat{a}_z$$

1.2 COORDINATE SYSTEMS

A coordinate system defines points of reference from which specific vector directions may be defined.

Depending on the geometry of the application, one coordinate system may lead to more efficient vector definitions than others. The three most commonly used co-ordinate systems used in the study of electromagnetics are rectangular coordinates (or Cartesian coordinates), cylindrical coordinates and spherical coordinates.

Note: An orthogonal system is one in which the coordinates are mutually perpendicular

1.2.1 Cartesian Coordinates

A vector at any point P is specified in terms of three mutually perpendicular components with unit vectors $\hat{a}_x, \hat{a}_y$ and $\hat{a}_z$. The unit vectors $\hat{a}_x, \hat{a}_y$ and $\hat{a}_z$ forms a right-handed set; i.e., turning $\hat{a}_x$ into $\hat{a}_y$ like a right handed screw, we move in the $\hat{a}_z$ direction.

A vector $\vec{A}$ in Cartesian (other wise known as rectangular) coordinates can be written as

(A_x, A_y, A_z) or $A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ ...(1.26)

Where $\hat{a}_x, \hat{a}_y, \hat{a}_z$ are unit vectors along the x, y and z directions

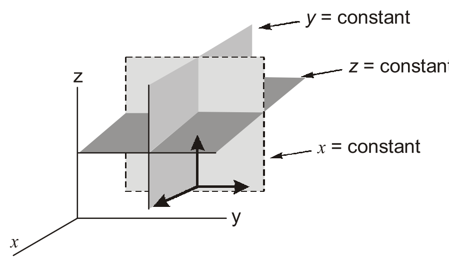


***Fig.:** A point in Cartesian coordinates is defined by the intersection of the three planes: x = constant, y = constant, z = constant. The three unit vectors are normal to each of the three surfaces.*

The ranges of the variables are:

$$-\infty \le x \le +\infty \quad ...(1.27\ a)$$
$$-\infty \le y \le +\infty \quad ...(1.27\ b)$$
$$-\infty \le z \le +\infty \quad ...(1.27\ c)$$

1.2.2 Cylindrical Coordinates

The cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point P in cylindrical coordinates is represented as (ρ, ϕ, z). Observe closely and note how we define each space variable; ρ is the radius of the cylinder passing through P or the radial distance from the z-axis; ϕ, called the azimuthal angle, is measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system. The ranges of the variables are:

$$0 \le \rho \le \infty \quad ...(1.28)$$
$$0 \le \phi \le 2\pi \quad ; \quad -\infty \le z \le +\infty$$

A vector $\vec{A}$ in cylindrical coordinates can be written as

(A_ρ, A_ϕ, A_z) or $A_\rho\hat{a}_\rho + A_\phi\hat{a}_\phi + A_z\hat{a}_z$ ...(1.29)

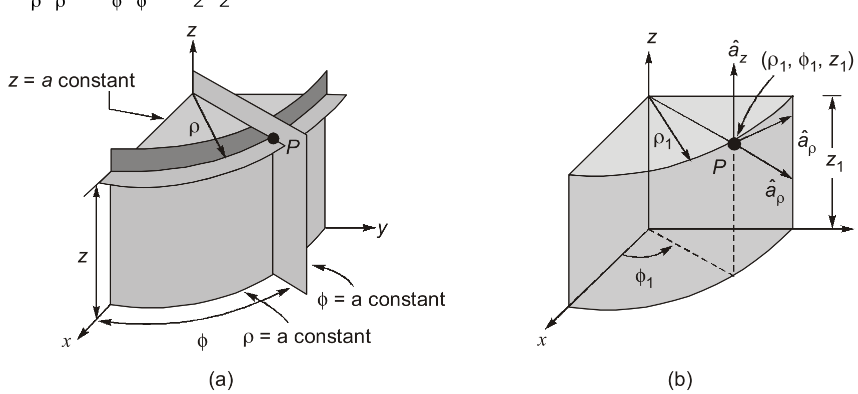


***Fig.:** (a) The point is defined by the intersection of the cylinder and the two planes. (b) Point P and unit vectors in the cylindrical coordinate system.*

Notice that the unit vectors $\hat{a}_\rho, \hat{a}_\phi$ and $\hat{a}_z$ are mutually perpendicular because our coordinate system is orthogonal.

$$\hat{a}_\rho \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_\rho = 0 \quad \text{...(1.30)}$$

$$\hat{a}_\rho \cdot \hat{a}_\rho = \hat{a}_\phi \cdot \hat{a}_\phi = \hat{a}_z \cdot \hat{a}_z = 1 \quad \text{...(1.31)}$$

$$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z \quad \text{...(1.32)}$$

$$\hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho \quad \text{...(1.33)}$$

$$\hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi \quad \text{...(1.34)}$$

1.2.3 Relationship between Cartesian and Cylindrical Coordinate System

The relationships between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system (ρ, ϕ, z) are easily obtained from figure below.

Point transformation,
$$\rho = \sqrt{x^2 + y^2}, \; \phi = \tan^{-1}\frac{y}{x}, \; z = z \quad \text{...(1.35)}$$

or,
$$x = \rho \cos \phi, \; y = \rho \sin \phi, \; z = z \quad \text{...(1.36)}$$

Whereas equation (1.35) is for transforming a point from Cartesian (x, y, z) to cylindrical (ρ, ϕ, z) coordinates, equation (1.36) is for $(\rho, \phi, z) \rightarrow (x, y, z)$ transformation.

The relationships between $\hat{a}_x, \hat{a}_y, \hat{a}_z$ and $\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z$ are

Vector transformation,
$$\hat{a}_x = \cos\phi \hat{a}_\rho - \sin\phi \hat{a}_\phi \quad \text{...(1.37 a)}$$

$$\hat{a}_y = \sin\phi \hat{a}_\rho + \cos\phi \hat{a}_\phi \quad \text{...(1.37 b)}$$

$$\hat{a}_z = \hat{a}_z \quad \text{...(1.37 c)}$$

or,
$$\hat{a}_\rho = \cos\phi \hat{a}_x + \sin\phi \hat{a}_y \quad \text{...(1.38 a)}$$

$$\hat{a}_\phi = -\sin\phi \hat{a}_x + \cos\phi \hat{a}_y \quad \text{...(1.38 b)}$$

$$\hat{a}_z = \hat{a}_z \quad \text{...(1.38 c)}$$

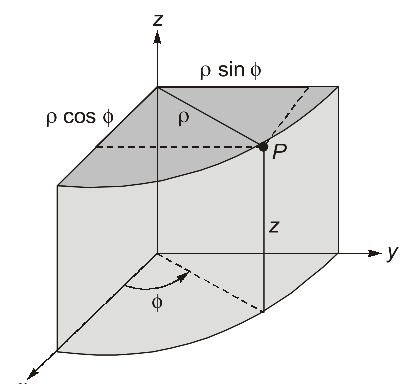


***Fig.:** Relationship between (x, y, z) and (ρ, ϕ, z)*

Finally, the relationship between (A_x, A_y, A_z) and (A_ρ, A_ϕ, A_z) are

$$\begin{vmatrix} A_\rho \\ A_\phi \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} \quad \text{...(1.39)}$$

$$\begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_\rho \\ A_\phi \\ A_z \end{vmatrix} \quad \text{...(1.40)}$$

1.2.4 Spherical Coordinates

The spherical coordinate system is most appropriate when dealing with problems having a degree of spherical symmetry. A point P can be represented as (r, θ, ϕ). From figure, we notice that r is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through P; θ (called the colatitudes) is the angle between the z-axis and the position vector of P; and ϕ is measured from the x-axis (the same azimuthal angle in cylindrical coordinates). According to these definitions, the ranges of the variables are

$$0 \le r \le \infty \quad \text{...(1.41)}$$

$$0 \le \theta \le \pi \quad ; \quad 0 \le \phi \le 2\pi$$

A vector $\vec{A}$ in spherical coordinates can be written as

(A_r, A_θ, A_ϕ) or $A_r \hat{a}_r + A_\theta \hat{\phi}_\theta + A_\phi \hat{a}_\phi$ ...(1.42)

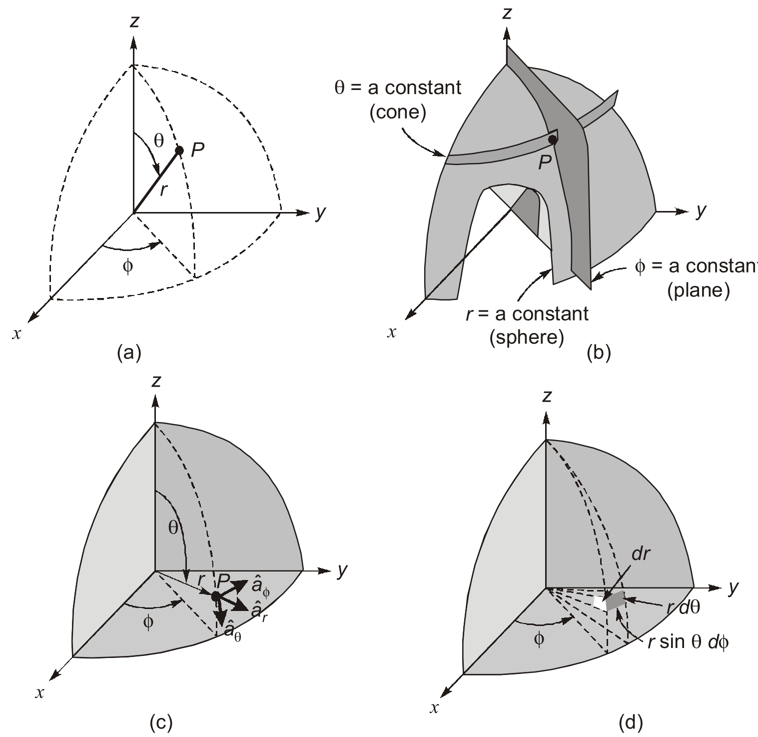


Fig.: *(a) Point P and unit vectors in the cylindrical coordinate system.*
(b) The three mutually perpendicular surfaces of the spherical coordinate system.
(c) The three unit vectors of spherical coordinates.
(d) The differential volume element in the spherical coordinate system.

Note: The unit vectors $\hat{a}_r$, $\hat{a}_\theta$, and $\hat{a}_\phi$ are mutually perpendicular because our coordinate system is orthogonal.

$$\hat{a}_r \cdot \hat{a}_\theta = \hat{a}_\theta \cdot \hat{a}_\phi = \hat{a}_\phi \cdot a_r = 0 \qquad \text{...(1.43)}$$

$$\hat{a}_r \cdot \hat{a}_r = \hat{a}_\theta \cdot a_\theta = \hat{a}_\phi \cdot \hat{a}_\phi = 1 \qquad \text{...(1.44)}$$

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi \qquad \text{...(1.45)}$$

$$\hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r \qquad \text{...(1.46)}$$

$$\hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta \qquad \text{...(1.47)}$$

1.2.5 Relationship between Cartesian Coordinate System and Spherical Coordinate System

The relationship between the variables (x, y, z) of the Cartesian coordinate system and those of the spherical coordinate system (ρ, θ, ϕ) are easily obtained from figure 1.8.

Point transformation, $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$, $\phi = \tan^{-1}\frac{y}{x}$ ...(1.48)

or $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$ ...(1.49)

The relationship between $\hat{a}_x, \hat{a}_y, \hat{a}_z$ and $\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$ are

$$\hat{a}_x = \sin\theta\cos\phi\hat{a}_r + \cos\theta\cos\phi\hat{a}_\theta - \sin\phi\hat{a}_\phi \quad ...(1.50\ a)$$

$$\hat{a}_y = \sin\theta\sin\phi\hat{a}_r + \cos\theta\sin\phi\hat{a}_\theta + \cos\phi\hat{a}_\phi \quad ...(1.50\ b)$$

$$\hat{a}_z = \cos\phi\hat{a}_r - \sin\phi\hat{a}_\phi \quad ...(1.50\ c)$$

or,

$$\hat{a}_r = \sin\theta\cos\phi\hat{a}_x + \sin\theta\sin\phi\hat{a}_y + \cos\theta\hat{a}_z \quad ...(1.51\ a)$$

$$\hat{a}_\theta = \cos\theta\cos\phi\hat{a}_x + \cos\theta\sin\phi\hat{a}_y - \sin\theta\hat{a}_z \quad ...(1.51\ b)$$

$$\hat{a}_\phi = -\sin\phi\hat{a}_x + \cos\phi\hat{a}_y \quad ...(1.51\ c)$$

Finally, the relationship between (A_x, A_y, A_z) and (A_r, A_θ, A_ϕ) are

Vector transformation,

$$\begin{vmatrix} A_r \\ A_\theta \\ A_\phi \end{vmatrix} \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} \quad ...(1.52)$$

or,

$$\begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} \begin{vmatrix} A_r \\ A_\theta \\ A_\phi \end{vmatrix}$$

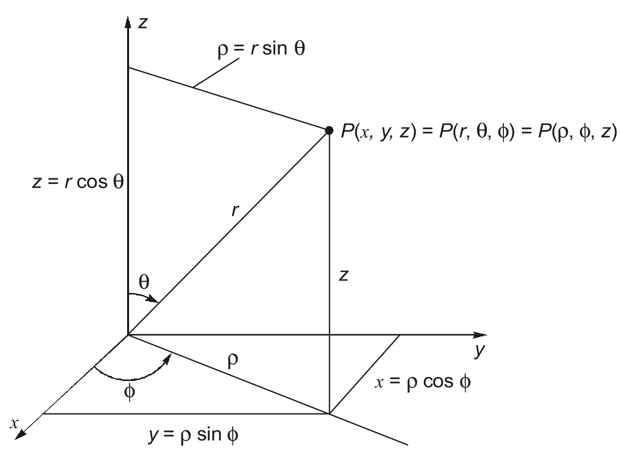


Fig.: *Relationships between space variables (x, y, z), (r, θ, ϕ) and (ρ, ϕ, z)*

	Cartesian (x, y, z)	Cylindrical (ρ, ϕ, z)	Spherical (r, θ, ϕ)
Cartesian (x, y, z)		$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$
Cylindrical (ρ, ϕ, z)	$\rho = \sqrt{x^2 + y^2}$ $\phi = \tan^{-1}\frac{y}{x}$ $z = z$		$\rho = r\sin\theta$ $\phi = \phi$ $z = r\cos\theta$
Spherical (r, θ, ϕ)	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$ $\phi = \tan^{-1}\frac{y}{x}$	$r = \sqrt{\rho^2 + z^2}$ $\theta = \tan^{-1}\frac{\rho}{z}$ $\phi = \phi$	

Table : Relationship between Different Sets

EXAMPLE : 1.3

Given point P (–2, 6, 3) and vector $\vec{A} = y\hat{a}_x + (x + z)\hat{a}_y$, express P and $\vec{A}$ in cylindrical and spherical coordinates. Evaluate A at P in the Cartesian, cylindrical, and spherical systems.

Solution:

At point P: $x = -2, y = 6, z = 3$. Hence,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4 + 36} = 6.32$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2} = 108.43°$$

$$z = 3$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4 + 36 + 9} = 7$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1}\frac{\sqrt{4 + 36}}{3} = 64.62°$$

OBJECTIVE BRAIN TEASERS

Q.1 A point is represented in Cartesian coordinates as $P(3, 4, 5)$, the radial component ρ in cylindrical coordinates will be
(a) less than (b) greater than
(c) equal to (d) unrelated to
r in spherical coordinates.

Q.2 Consider a closed surface S surrounding a volume V. If $\vec{r}$ is the position vector of a point inside S, with $\hat{n}$ the unit normal on S, the value of the integral $\oiint_S 2\vec{r}\cdot\hat{n}\,dS$ is
(a) 3 V (b) 2 V
(c) 6 V (d) 4 V

Q.3 Consider a vector $\vec{E} = z\hat{a}_x + (x + y)\hat{a}_y$, the z component of the vector in cylindrical coordinates will be
(a) z
(b) $z\cos\phi + (x + y)\sin\phi$
(c) $-z\sin\phi + (x + y)\cos\phi$
(d) zero

Q.4 The direction of vector $\vec{A}$ is radially outward from the origin, with $|A| = kr^n$ where $r^2 = x^2 + y^2 + z^2$ and k is a constant. The value of n for which $\nabla\cdot\vec{A} = 0$ is
(a) – 2 (b) 2
(c) 1 (d) 0

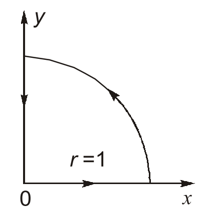
Q.5 Let a point in spherical and cylindrical coordinates are (r, θ, ϕ) and (ρ, ϕ, z). The radial component r in spherical coordinates is related to components in cylindrical coordinates as
(a) ρ (b) $\rho\cos\phi$
(c) $z\tan^{-1}\phi$ (d) $(\rho^2 + z^2)^{1/2}$

Q.6 Given the vector

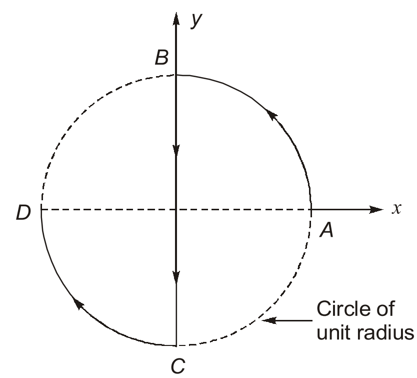
$$\vec{A} = (\cos x)(\sin y)\hat{a}_x + (\sin x)(\cos y)\hat{a}_y,$$

where $\hat{a}_x, \hat{a}_y$ denote unit vectors along x, y directions, respectively. The magnitude of curl of $\vec{A}$ is
(a) 0 (b) 1
(c) –1 (d) 2

Q.7 Given a vector field $\vec{A} = 2r\cos\phi\,\hat{a}_r$ in cylindrical coordinates. For the contour as shown below, $\oint \vec{A}\cdot d\vec{l}$ is
(a) 1
(b) $1 - (\pi/2)$
(c) $1 + (\pi/2)$
(d) – 1



Q.8



What is the value of the integral $\int_c d\vec{l}$ along the curve c (c is the curve ABCD in the direction of the arrow)?
(a) $2(\hat{a}_x + \hat{a}_y)/\sqrt{2}$
(b) $-2(\hat{a}_x + \hat{a}_y)/\sqrt{2}$
(c) $2\hat{a}_x$
(d) $-2\hat{a}_y$

ANSWERS KEY

1. (a) **2.** (c) **3.** (d) **4.** (a) **5.** (d)
6. (a) **7.** (a) **8.** (d)

HINTS & EXPLANATIONS

1. (a)

$P(3, 4, 5)$, $\rho = \sqrt{3^2 + 4^2} = 5$

$r = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$

So, $r > \rho$

CONVENTIONAL BRAIN TEASERS

Q.1 Find the area of the curved surface of a right circular cylinder of radius r and height h using cylindrical coordinates.

1. (Sol.)

radius of cylinder, 'r'; height of cylinder, 'h'

Curved surface area,
$$A = \int_0^{2\pi}\int_0^{h} r\, dz\, d\phi = r\int_0^{2\pi} h\, d\phi = 2\pi rh$$

Q.2 Calculate the volume in spherical coordinates defined by

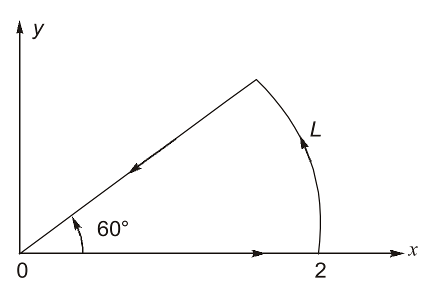
$$1 \le r \le 2 \text{ m},\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2}$$

2. (Sol.)

$$\text{Volume, } V = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^{2} r^2 \sin\theta\, dr\, d\theta\, d\phi = \int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_1^2 \sin\theta\, d\theta\, d\phi$$

$$V = \frac{7}{3}\int_0^{\pi/2}[-\cos\theta]_0^{\pi/2}\, d\phi = \frac{7}{3}[\phi]_0^{\pi/2} = \frac{7}{6}\pi \text{m}^3$$

Q.3 Calculate the circulation of $A = \rho\cos\phi\,\hat{a}_\rho + z\sin\phi\,\hat{a}_z$ around the edge L of the wedge defined by $0 < \rho < 2$, $0 \le \phi \le 60°$, $z = 0$ and shown in figure.



3. (Sol.)

$$A = \rho\cos\phi\,\hat{a}_\rho + z\sin\phi\,\hat{a}_z$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{\rho}\begin{bmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \rho\cos\phi & 0 & z\sin\phi \end{bmatrix} = \frac{1}{\rho}[(z\cos\phi)\hat{a}_\rho - \rho\hat{a}_\phi[0] + (\rho\sin\phi)\hat{a}_z]$$

$$= \frac{z\cos\phi}{\rho}\hat{a}_\rho + \frac{\rho\sin\phi}{\rho}\hat{a}_z$$